3. MÖBIUS TRANSFORMATIONS AND THE EXTENDED COMPLEX PLANE

It can be helpful, particularly with the Möbius Transformations we are about to meet, to include a further point ∞ to the complex plane.

Definition 40 *The* ***extended complex plane*** *is the set* $\mathbb{C} \cup \{\infty\}$ *and we denote this by* $\tilde{\mathbb{C}}$.

Definition 41 *A* ***Möbius transformation*** *is a map* $f : \tilde{\mathbb{C}} \to \tilde{\mathbb{C}}$ *of the form*

$$f(z) = \frac{az+b}{cz+d} \qquad \text{where } ad \neq bc.$$

We define

$$f(\infty) = \begin{cases} \frac{a}{c} & \text{if } c \neq 0, \\ \infty & \text{if } c = 0, \end{cases}$$

and also $f(-d/c) = \infty$ *if* $c \neq 0$.

Remark 42 *The Möbius transfromations clearly have a "matrix" look to them and there are good reasons for this, though the full reason is beyond the scope of this course. The extended complex plane is the* ***complex projective line*** $P(\mathbb{C}^2)$ *and the Möbius transformations are its* ***projective transformations*** $PGL(2, \mathbb{C})$.

The complex projective line is as follows. Suppose that we consider the equivalence classes of non-zero complex pairs $\mathbb{C}^2 \setminus \{(0,0)\}$ *under the equivalence relation*

$$(z_1, w_1) \sim (z_2, w_2) \text{ if and only there is a non-zero } \lambda \in \mathbb{C}\setminus\{0\} \text{ such that } z_1 = \lambda w_1, z_2 = \lambda w_2.$$

Then each equivalence class has a representative of the form $(z, 1)$ *where* $z \in \mathbb{C}$ *except* $(1, 0)$. *The former can be thought of as* z *and the latter as* ∞.

Matrices don't act on these pairs in a well-defined way, but equivalence classes of invertible matrices do. If we identify 2×2 *complex matrices under the equivalence relation*

$$M_1 \sim M_2 \text{ if and only if there is a non-zero } \lambda \in \mathbb{C}\setminus\{0\}, \text{ such that } M_1 = \lambda M_2$$

then the Möbius transformations can be identified with the equivalence classes of invertible matrices which is denoted $PGL(2, \mathbb{C})$ *a* ***P****rojective* ***G****eneral* ***L****inear group. The Möbius transformation*

$$\frac{az+b}{cz+d} \text{ is then identified with the equivalence class of } \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Proposition 43 *Möbius transformations form the group of transformations $\tilde{\mathbb{C}} \to \tilde{\mathbb{C}}$ generated (under composition) by:*

- ***translations*** *— maps of the form $z \mapsto z + k$ where $k \in \mathbb{C}$;*
- ***scalings*** *or* ***dilations****— maps of the form $z \mapsto kz$ where $k \in \mathbb{C} \setminus \{0\}$;*
- ***inversion*** *— the map $z \mapsto 1/z$.(Note this map is not an actual inversion in the sense of inverting in a circle.)*

Proof. Note that if $c \neq 0$ then

$$\frac{az+b}{cz+d} = \frac{a}{c} - \frac{bc-ad}{c^2z+dc}$$

is a composition of various translations, scalings and inversion. If $c = 0$ then $d \neq 0$ and clearly $z \mapsto (a/d)\,z + (b/d)$ is a composition of a scaling and a translation.

This shows that the set of Möbius transformations are a subset of the group generated by translations, scalings and inversion. It is also clear that translations, scalings and inversion are all special types of Möbius transformations. Further if $f(z) = (az+b)/(cz+d)$ is a Möbius transformation then

$$\begin{aligned} f(z+k) &= \frac{az+(ak+b)}{cz+(ck+d)} \text{ where } a(ck+d) - c(ak+b) = ad-bc \neq 0; \\ f(kz) &= \frac{akz+b}{ckz+d} \text{ where } (ak)\,d - b\,(ck) = k\,(ad-bc) \neq 0 \text{ if } k \neq 0; \\ f\left(\frac{1}{z}\right) &= \frac{bz+a}{dz+c} \text{ where } bc-ad \neq 0. \end{aligned}$$

Hence Möbius transformations composed with these generators yield further Möbius transformations and the result follows. ■

Proposition 44 *The Möbius transformations are bijections on $\tilde{\mathbb{C}}$.*

Proof.

- The translation $z \mapsto z+k, \infty \mapsto \infty$ is clearly a bijection with inverse $z \mapsto z-k, \infty \mapsto \infty$.
- The scaling $z \mapsto kz, \infty \mapsto \infty$ is clearly a bijection with inverse $z \mapsto z/k, \infty \mapsto \infty$.
- And inversion $z \mapsto 1/z, 0 \mapsto \infty, \infty \mapsto 0$ is a bijection which is its own inverse.

Hence any composition of these maps, i.e. the Möbius transformations, are all bijections. ■

Definition 45 *We will use the term* ***circline*** *to denote anything which is a circle or a line in the complex plane. (It will be helpful, in much of this course, to think of a line as a circle that happens to go through ∞.)*

Note from Proposition 34 that the circlines are the solutions sets of the equations

$$Az\bar{z} + \bar{B}z + B\bar{z} + C = 0 \tag{3.1}$$

where $A, C \in \mathbb{R}$ and $B \in \mathbb{C}$.

Proposition 46 *The Möbius transformations map circlines to circlines.*

Proof. In Corollary 37 we showed that inversion $z \mapsto 1/z$ maps circlines to circlines, and it is clear that translations also map circlines to circlines. Also the image of (3.1) under the scaling $z \mapsto kz$ has equation

$$Az\bar{z} + \overline{Bk}z + Bk\bar{z} + |k|^2 C = 0$$

which is another circline. Hence a Möbius transformation, which can be written as a composition of these maps, also maps circlines to circlines. ■

Example 47 *Show that the maps*

$$f(z) = \frac{e^{i\theta}(z-a)}{1-\bar{a}z} \quad \text{where } \theta \in \mathbb{R} \text{ and } |a| < 1$$

maps the unit circle $|z| = 1$ to itself and a to 0.

Proof. Clearly a maps to 0. Suppose now that $|z| = 1$. Then

$$|f(z)| = \left|\frac{e^{i\theta}(z-a)}{1-\bar{a}z}\right| = \frac{|z-a|\,|\bar{z}|}{|1-\bar{a}z|} = \frac{|1-a\bar{z}|}{|1-\bar{a}z|} = 1$$

as the numerator is the conjugate of the denominator. ■

Exercise 17 *Show that any Möbius transformation map which maps the unit circle $|z| = 1$ to itself and the interior to the interior has the above form for some θ and a.*

Example 48 *Find a Möbius transformation which maps the unit circle $|z| = 1$ to the real axis and the real axis to the imaginary axis.*

Solution. Initially it might well seem that the two diagrams — a circle divided by a line and two perpendicular lines meeting at the origin — are very different. The circle meets the real axis at right angles at two points, but in the second diagram this intersection is still there but "hidden" at infinity. So if we take one of these intersections, say 1, to ∞ and the other, -1, to the origin the diagrams will look much more similar. Let's start then with

$$f(z) = \frac{z+1}{z-1}.$$

As required f takes -1 to 0 and 1 to ∞. Also $f(0) = -1$ and so f takes the real axis to a circline containing $0, -1, \infty$, that is the real axis (rather than the imaginary axis). It takes the unit circle to a circle containing $0, \infty$ and $f(i) = (i+1)/(i-1) = -i$, that is the imaginary axis (rather than the real axis). To unmuddle our images we can rotate by $\pi/2$ about the origin — hence we see the map

$$z \mapsto i\left(\frac{z+1}{z-1}\right)$$

solves the given problem. ■

Proposition 49 *The Mobius transformations are* ***conformal.*** *That is, they preserve angles, together with their orientation.*

Proof. It is enough to consider how the real axis meets with a general line through 1, and to verify that translations, scalings and inversion are all conformal at this point. Clearly translations, as they are isometries, preserve angles.

Suppose now that we have the real axis and the line $z = 1 + re^{i\theta}$ where r is an arbitrary real number. Under the map $z \mapsto kz$, where $k \neq 0$, then the real axis maps to the line through 0 and k — further the lines' intersection, formerly at 1, now maps to k. The second line maps to the points $k + kre^{i\theta}$. The angle they now make is

$$\arg\left(\frac{\left(k + ke^{i\theta}\right) - k}{k}\right) = \theta,$$

the same as it was previously.

Finally inversion takes the real axis to itself, albeit in a reverse fashion, and maps 1 to itself. Further the point $1 + re^{i\theta}$ maps to $\left(1 + re^{i\theta}\right)^{-1}$. Then the angle this curve (which is a circle, though not obviously so!), makes with 1 can be determined from the tangent vector

$$\lim_{r \to 1} \frac{1}{r}\left(\frac{1}{1 + re^{i\theta}} - 1\right) = \lim_{r \to 0}\left(\frac{-e^{i\theta}}{1 + re^{i\theta}}\right) = -e^{i\theta}.$$

The image of the line $\arg(z - 1) = \theta$ is now pointing back and down at angle θ and the image of the real axis is pointings backwards from 1, so measuring the angle with the same orientation it is still θ. Hence $z \mapsto 1/z$ is also conformal. Finally we see any composition of conformal maps will be conformal and so the Möbius transformations preserve angles, together with their sense. ■

Proposition 50 *Given three distinct points $z_1, z_2, z_3 \in \tilde{\mathbb{C}}$ and three other distinct points $w_1, w_2, w_3 \in \tilde{\mathbb{C}}$ then there is a unique Möbius transformation f such that $f(z_i) = w_i$ for $i = 1, 2, 3$.*

Proof. Note that the map

$$f(z) = \frac{(z_2 - z_3)(z - z_1)}{(z_2 - z_1)(z - z_3)}$$

is a Möbius transformation which maps z_1, z_2, z_3 to $0, 1, \infty$. There is a similar Möbius transformation g which sends w_1, w_2, w_3 to $0, 1, \infty$. Hence $g^{-1}f$ is a Möbius transformation which maps each z_i to w_i.

To show uniqueness suppose h is another such map. Then ghf^{-1} is a Möbius transformation which maps $0, 1, \infty$ to $0, 1, \infty$. If we write

$$ghf^{-1}(z) = \frac{az + b}{cz + d}$$

then $0 \mapsto 0$ means $b = 0$, $1 \mapsto 1$ means $a + b = c + d$ and $\infty \mapsto \infty$ means $c = 0$. Hence $ghf^{-1}(z) = z$ for all z and $h = g^{-1}f$ as required. ■

Theorem 51 *Let $f : \tilde{\mathbb{C}} \to \tilde{\mathbb{C}}$ be a Möbius transformation, not equal to the identity. Then either*

(i) f fixes one point in $\tilde{\mathbb{C}}$ and is similar to a translation, or

(ii) f fixes two points in $\tilde{\mathbb{C}}$ and is similar to a scaling.

Proof. Let $f(z) = (az+b)/(cz+d)$. Note that f fixes ∞ if and only if $c = 0$. A point $z \in \mathbb{C}$ is fixed by f if and only if.

$$cz^2 + (d-a)z - b = 0. \tag{3.2}$$

Let $\Delta = (d-a)^2 + 4bc$ denote the discriminant of this quadratic.

(a) If $c = 0$ and $a = d \neq 0$ then (3.2) has no solutions and so f only fixes ∞. In this case we see that $f(z) = z + b/d$ is clearly a translation.

(b) If $c = 0$ and $a \neq d$ then (3.2) has one solutions $z = b/(d-a)$ and so f only fixes two points. If we move the origin to $b/(d-a)$ by means of the translation

$$g(z) = z - \frac{b}{d-a}$$

then we see

$$gf(z) = \frac{az+b}{d} - \frac{b}{d-a} = \frac{(d-a)az - ab}{d(d-a)} = \frac{a}{d}\left(z - \frac{b}{d-a}\right) = \frac{a}{d}g(z)$$

showing that $gfg^{-1}(z) = (a/d)z$ is a scaling.

(c) If $c \neq 0$ then (3.2) has two solutions which we'll denote as α and β. We now set

$$g(z) = \frac{z-\alpha}{z-\beta}.$$

Then

$$\begin{aligned} gf(z) &= \frac{\frac{az+b}{cz+d} - \alpha}{\frac{az+b}{cz+d} - \beta} = \frac{az+b-\alpha cz - \alpha d}{az+b-\beta cz - \beta d} \\ &= \left(\frac{a-\alpha c}{a-\beta c}\right)\frac{z + \left(\frac{b-\alpha d}{a-\alpha c}\right)}{z + \left(\frac{b-\beta d}{a-\beta c}\right)}. \end{aligned}$$

But we can easily see that

$$\frac{b-\alpha d}{a-\alpha c} = -\alpha \quad \text{and} \quad \frac{b-\beta d}{a-\beta c} = -\beta$$

as α and β are roots of (3.2). So

$$gf(z) = k\left(\frac{z-\alpha}{z-\beta}\right) = kg(z)$$

where $k = (a-\alpha c)/(b-\beta c)$ and hence $gfg^{-1}(z) = kz$ is a scaling as required. ■